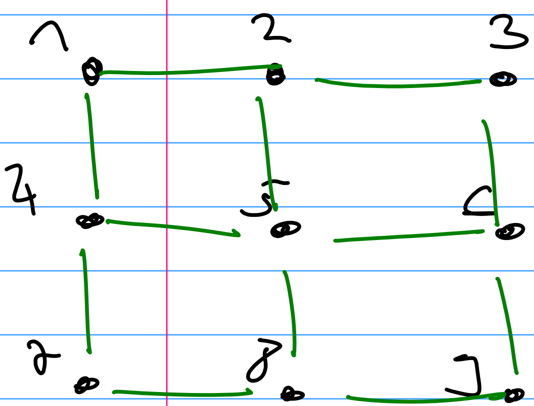
McKay's algorithm for isomorphism checking



Obvious idea:

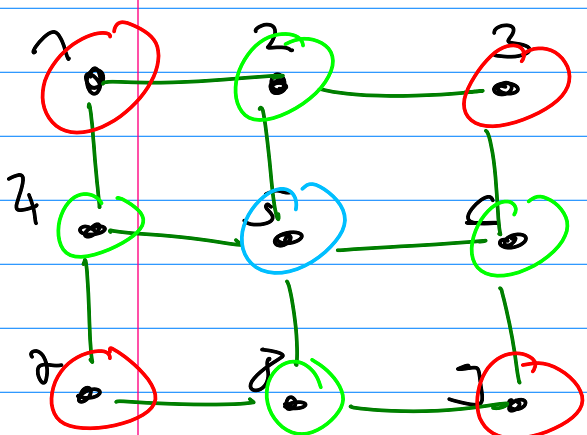
Order nodes by degrees

$$\underbrace{(1\,3\,7\,9\,|\,2\,4\,6\,8\,|\,5)}_{\text{ordered partition } \pi}$$

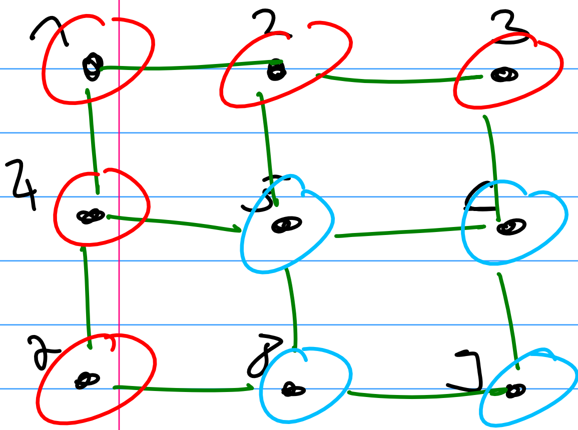
Formally $\pi = (V_1, V_2, \dots, V_r)$, ordered by degree.

π is equitable, iff

$$\forall\, 1 \le i,j \le r\ \forall v,w \in V_i : \deg(v, V_j) = \deg(w, V_j)$$



$\pi = (1379|2468|5)$ is equitable



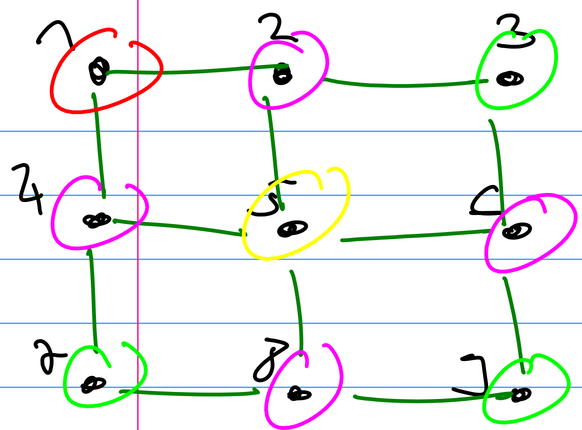
$\pi = (1\,2\,3\,4\,7 \mid 5\,6\,8\,9)$ is not equitable

We say V_j <u>shatters</u> V_i if
$\exists\, v, w \in V_i$ such that $\deg(v, V_j) \neq \deg(w, V_j)$
Let $D = \{(i,j) \mid V_j \text{ shatters } V_i\}$

Example

$\pi = (1\,2\,3\,4\,5\,6\,7\,8\,9) = (V_1)$

V_1 shatters V_1, $B = \{(1,1)\}$



$$\pi = (\overset{V_1}{1} \mid \overset{V_2}{3\,7\,9} \mid \overset{V_3}{2\,4\,6\,8} \mid \overset{V_4}{5})$$

is not equitable

$B = \{(3,1), (3,2)\}$

A <u>shattering</u> of V_i by V_j is the ordered partition $(X_1, \ldots, X_t)$ of V_i, such that if $v \in X_k$ and $w \in X_\ell$ then $k < \ell$ iff $\deg(v, V_j) < \deg(w, V_j)$

$(X_1, \ldots, X_t)$ sorts the vertices of V_i by their degree to V_j.

$$\pi = (1 \mid 3\,7\,9 \mid 2\,4\,6\,8 \mid 5)$$

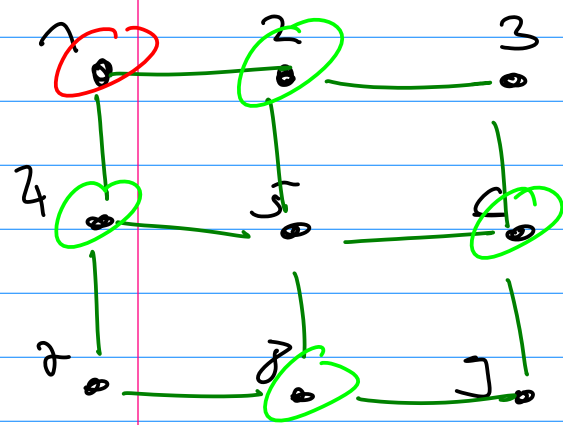
As $(3,1) \in B$:

1 shatters 2468

$\pi = (1 \mid 3\,7\,9 \mid 2\,4\,6\,8 \mid 5)$

1 shatters 2468

As $(3,1) \in \mathcal{B}$:



a new partition

$(1 \mid \underline{3\,7\,9} \mid 6\,8 \mid \underline{2\,4} \mid 5)$

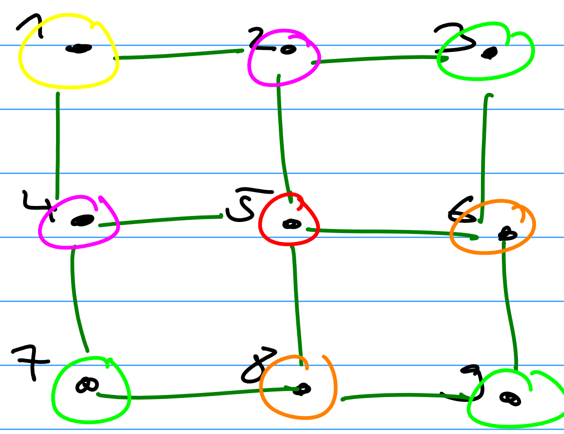
$V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5$

iterate!

$(1 \mid \underline{3\,7\,9} \mid 6\,8 \mid \underline{2\,4} \mid 5)$

$V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5$

24 shatters 379



$\mathcal{B} = \{(2,4)\}$

$\rightsquigarrow \quad (1 \mid 9 \mid 3\,7 \mid 6\,8 \mid 2\,4 \mid 5)$

"equitable refinement"

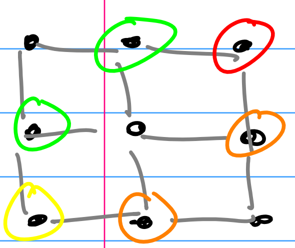
The search tree $(T(G))$

$(1379|2468|5)$

$(1|9|37|68|24|5)$

v_1 v_2 v_3 v_4 v_5 v_6 v_7

$(1|9|3|7|8|6|4|2|5)$ $(1|9|7|3|6|8|2|4|5)$

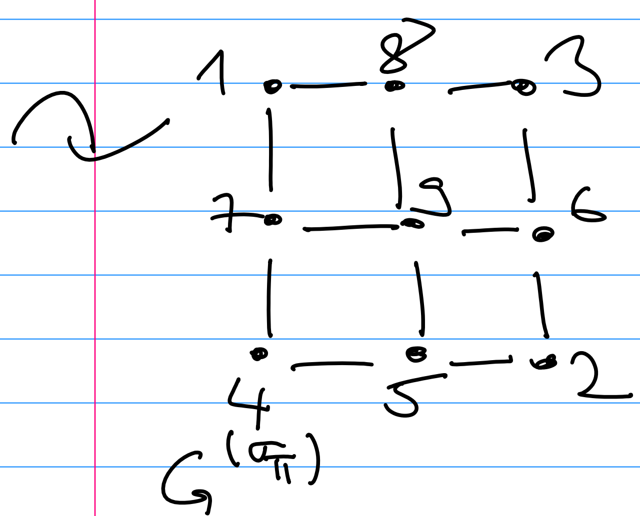


$B = \{(6,3), (5,3), (6,4), (5,4)\}$

$(1|9|3|7|68|4|2|5)$

$B = \{(5,3), (5,4)\}$

$(1|9|3|7|8|6|4|2|5)$



$G^{(\sigma_\pi)}$

σ_π: $1 \to 1$
$9 \to 2$
$3 \to 3$
$7 \to 4$
$8 \to 5$
$6 \to 6$
$4 \to 7$
$2 \to 8$
$5 \to 9$

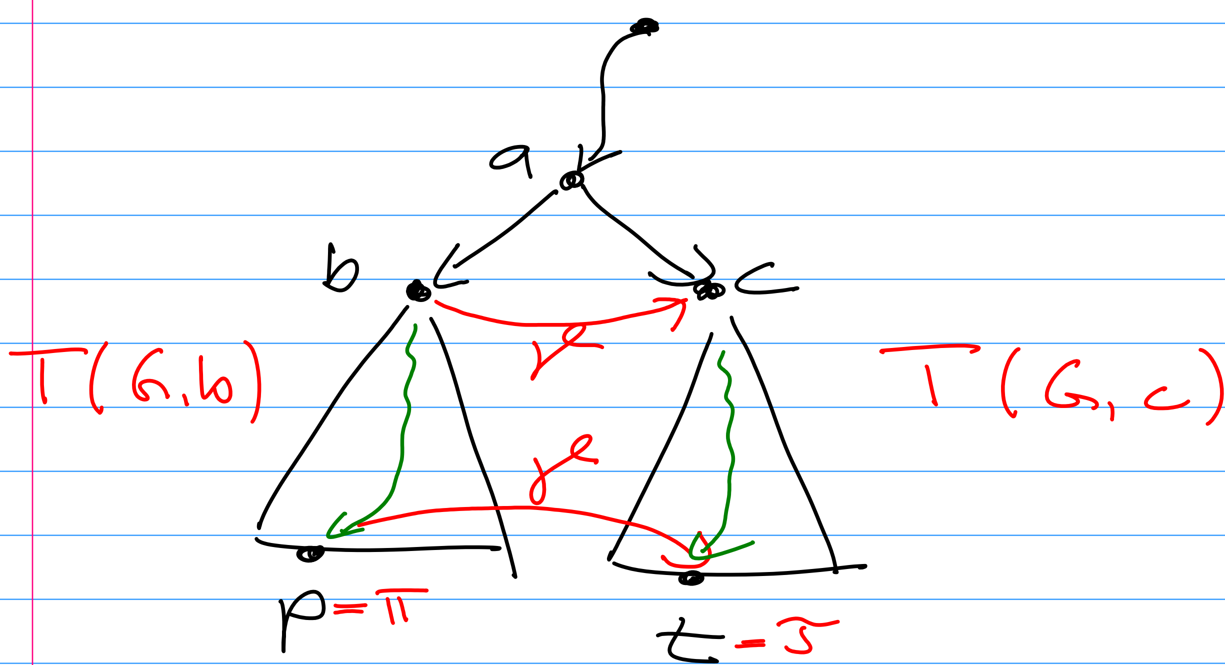
Canonical Isomorph Function (McKay):

$$C(G) = \max\{G^{(\sigma_\pi)} \mid \pi \text{ is a leaf of } T(G)\}$$

paper page 8: <u>all</u> leaves are the same (this is not true in general!)

⇝ the graph as depicted above is <u>the</u> canonical isomorph.

<u>Improvements</u> (clever pruning)



consider permutation $\gamma = \sigma_\pi(\sigma_\varsigma^{-1})$

Assume γ is an automorphism found via depth first traversal

⇝ $T(G,c)$ is isomorphic to $T(G,b)$ via γ, and so the set of terminal nodes

⇝ prune $T(G,c)$